B. STAT. + B. MATH

Ankur Roy Choudhury

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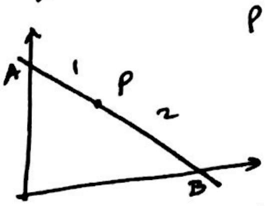
VOL-2

BSTAT & B. MATH (2012)

1) A rod AB of length 3 rests on a wall as follows:
P is a point on AB such that AP : PB = 1:2
If the rod slides along the wall, then locus of P lies on



a) $2x + y + xy = 2$ (b) $4x^2 + y^2 = 4$
c) $4x^2 + xy + y^2 = 4$ (d) $x^2 + y^2 - x - 2y = 0$.

Sol: Let coordinate of P be (h, k); $A(0, a)$; $B(b,)$.

AP : PB = 1:2. $\therefore$ $(h, k) = \left(\frac{1 \times b + 0 \times 2}{1+2}, \frac{a \times 2 + 1 \times 0}{1+2}\right)$

$\therefore (h, k) = \left(\frac{b}{3}, \frac{2a}{3}\right) \Rightarrow 3h = b$
$\frac{3k}{2} = a$

Now, $a^2 + b^2 = c^2 \Rightarrow \frac{9k^2}{4} + 9h^2 = c^2 \Rightarrow 4h^2 + k^2 = \frac{4c^2}{9}$

$\Rightarrow$ locus is $4x^2 + y^2 = c_1^2$ (Here c & c_1 is constant) as length of rod is constant). Hence option (b).

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2) Consider the equation $x^2 + y^2 = 2007$. How many solutions (x, y) exist such that x and y are positive integers?

✓(a) None (b) Exactly two (c) More than two but finitely many
(d) Infinitely many.

Sol: $x^2 + y^2 = 2007 \Rightarrow x^2 + y^2 = (\text{odd no}) \Rightarrow$ One of x & y is odd & another is even.

Let x is even and y is odd (without loss of generality);

Now $x^2 \equiv 0 \pmod 4$ (as x is even)
$y \equiv \pm 1 \pmod 4 \Rightarrow y^2 \equiv 1 \pmod 4$

Now dividing Equation by 4 we get

$x^2 + y^2 = 2007$

$\Rightarrow (0+1) \bmod 4 \equiv 3 \pmod 4$ Impossible, No solution.

$\Rightarrow 1 \equiv 3 \pmod 4$

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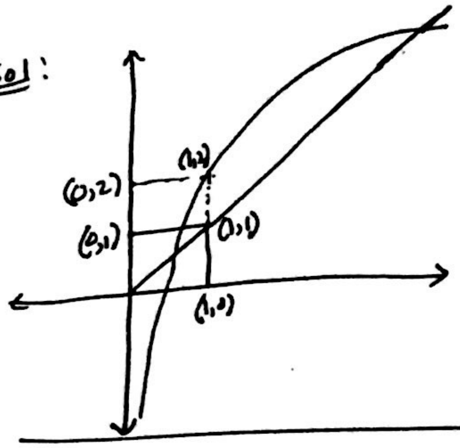
3) Consider the functions $f_1(x) = x$, $f_2(x) = 2 + \log_e x$, $x > 0$ (where e is the base of natural logarithm). The graphs of the functions intersect (a) once in $(0,1)$ and never in $(1, \infty)$

✓(b) once in $(0,1)$ and never in (e^2, ∞)

(c) once in $(0,1)$ and once in (e, e^2)

(d) more than twice in $(0, \infty)$.

Sol:



Clearly graphs meet once in $(0,1)$ and another in (e, e^2)

as $f_1(e) = e$ & $f_2(e) = 3 \Rightarrow f_1(e) < f_2(e)$

and $f_1(e^2) = \underset{\approx 7}{e^2}$ & $f_2(e^2) = 4 \Rightarrow f_1(e^2) > f_2(e^2)$

So option (b).

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4) Consider the sequence $u_n = \sum_{r=1}^{n} \frac{r}{2^r}$, $n \geq 1$

Then the limit of u_n as $n \to \infty$ is

a) 1 ✓b) 2 c) e d) 1/2

Sol: $u_n = 1/2 + 2/2^2 + 3/2^3 + \dots + n/2^n$ (A.G.P series)

$$\frac{1}{2}u_n = \quad 1/2^2 + 2/2^3 + \dots + \frac{n-1}{2^n} + \frac{n}{2^{n+1}}$$

$$\frac{u_n}{2} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

$$\Rightarrow \cancel{\tfrac{1}{2}}\, u_n = \cancel{\tfrac{1}{2}}\left[\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right) - \frac{n}{2^n}\right]$$

$$\Rightarrow u_n = \frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-1/2} - \frac{n}{2^n} = 2\left[1-\frac{1}{2^n}\right] - \frac{n}{2^n}$$

As $n \to \infty$, $\therefore \lim_{n\to\infty} u_n = \lim_{n\to\infty} 2\left[1-\frac{1}{2^n}\right] - \lim_{n\to\infty}\frac{n}{2^n}\left(\frac{\infty}{\infty}\right)$

$$\Rightarrow \lim_{n\to\infty} u_n = 2(1-0) - \lim_{n\to\infty}\frac{1}{n(2^{n-1})} \quad \text{(Apply L.H. rule)}$$

$$\Rightarrow u_n = 2 - 0 = 2.$$

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5) Suppose that z is any complex no which is not equal to any of $\{3, 3\omega, 3\omega^2\}$ where ω is a complex cube root of unity. Then $\frac{1}{z-3} + \frac{1}{z-3\omega} + \frac{1}{z-3\omega^2}$ equals

a) $\frac{3z^2+3z}{(z-3)^3}$ b) $\frac{3z^2+3\omega z}{z^3-27}$ c) $\frac{3z^2}{z^3-3z^2+9z-27}$ ✓d) $\frac{3z^2}{z^3-27}$

Sol $\Rightarrow$

$$\frac{(z-3\omega)(z-3\omega^2)+(z-3)(z-3\omega^2)+(z-3)(z-3\omega)}{(z-3)(z-3\omega)(z-3\omega^2)}$$

$$= \frac{(z^2-3\omega z-3\omega^2 z+9\omega^3) + (z-3)[z-3\omega^2+z-3\omega]}{(z-3)[z^2-3\omega z-3\omega^2 z+9\omega^3]}$$

$$= \frac{[z^2-3z(\omega+\omega^2)+9] + (z-3)[2z-3(\omega+\omega^2)]}{(z-3)[z^2-3z(\omega+\omega^2)+9]} = \frac{(z^2+3z+9)+(z-3)(2z+3)}{(z-3)[z^2+3z+9]}$$

$$= (z^2+3z+9+2z^2+3z-6z-9)/(z^3-27) = \frac{3z^2}{z^3-27}.$$

6) Consider all functions $f : \{1,2,3,4\} \to \{1,2,3,4\}$ which are one-one, ONTO and satisfy the following property:
if $f(K)$ is odd then $f(K+1)$ is even, $K = 1, 2, 3$
The no. of such functions is

a) 4 b) 8 ✓c) 12 d) 16.

Sol: $\{K, f(K)\} = \{(1,1), (2,2), (3,3), (4,4)\}$; $\{(1,1), (2,2), (3,4), (4,3)\}$;
$\{(1,1), (2,4), (3,3), (4,2)\}$, $\{(1,1), (2,4), (3,2), (4,3)\}$;
$\{(1,2), (2,1), (3,4), (4,3)\}$, $\{(1,2), (2,3), (3,4), (4,1)\}$;
$\{(1,3), (2,2), (3,1), (4,4)\}$; $\{(1,3), (2,2), (3,4), (4,1)\}$,
$\{(1,3), (2,4), (3,1), (4,2)\}$; $\{(1,3), (2,4), (3,2), (4,1)\}$,
$\{(1,4), (2,1), (3,2), (4,3)\}$, $\{(1,4), (2,3), (3,2), (4,1)\}$,

No. of such functions is 12.

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7) A function $f : R \to R$ is defined by $f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \le 0 \end{cases}$
Then,

a) f is not continuous
b) f is differentiable but f' is not continuous
c) f is continuous but $f'(0)$ does not exist
✓d) f is differentiable and f' is continuous.

Sol: Clearly f is continuous.

$f'(x) = \begin{cases} \dfrac{e^{-1/x}}{x^2} & x > 0 \\ 0 & x \le 0 \end{cases}$

$f'(0^+) = \lim_{x\to 0^+} \dfrac{e^{-1/x}}{x^2} \left(\dfrac{\infty}{\infty}\right)$

$= \lim_{x\to 0^+} \dfrac{+2/x^3}{e^{+1/x} x^{-1}/x^2} = \lim_{x\to 0^+} \dfrac{-2/x^2}{e^{1/x} x^{-1}/x^2} = 0$

$\therefore$ Also $f'(0^-) = 0 \Rightarrow f'(0)$ exist $\Rightarrow f(x)$ is differentiable on $x \in R$. Also $f'(x)$ is continuous at $x = 0$.

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8) The last digit of $9! + 3^{9966}$ is

(a) 3 ✓(b) 9 c) 7 d) 1

Sol: last digit of $9! = 0$

now: $3^2 \equiv (-1) \pmod{10} \Rightarrow (3^2)^{4983} \equiv (-1)^{4983} \pmod{10} \equiv -1 \pmod{10}$
$\equiv 9 \pmod{10}$.

$\therefore$ last digit is 9.

Them,

a) maximum of f is attained inside the interval $(2,3)$

b) minimum of f is $28/5$

✓c) maximum of f is $28/5$

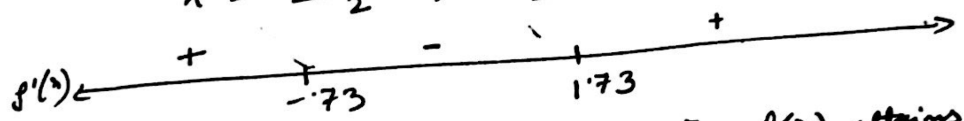
d) f is a decreasing function in $(2,3)$.

Sol: $f'(x) = \dfrac{\{(4x+3)(2x-1) - (2x^2+3x+1)\times 2\}}{(2x-1)^2} = 0$

$\Rightarrow (4x+3)(2x-1) - (2x^2+3x+1)2 = 0 \Rightarrow 4x^2-4x-5=0$

$\Rightarrow x = \dfrac{4 \pm \sqrt{16+80}}{8} = \dfrac{4\pm\sqrt{6}}{8} = \dfrac{1\pm\sqrt{6}}{2}$

$x = \dfrac{1+2.45}{2}, \dfrac{1-2.45}{2} \Rightarrow x = 1.73, -.73 \therefore f'(x) = \dfrac{(x-1.73)(x+.7}{(2x-1)^2}$



So for no value of x in $[2,3]$, $f(x)$ attains max/min value.

$f(x)$ is not decreasing in $(2,3)$.

So $f(2) = 5$, $f(3) = 28/5$.

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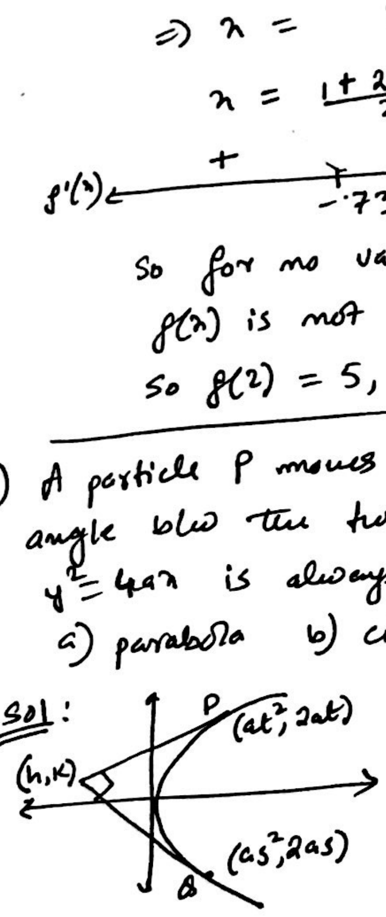
w) A particle P moves in the plane in such a way that the angle b/w the two tangents drawn from P to the curve $y^2=4ax$ is always $90°$. The locus of P is

a) parabola b) circle c) ellipse d) St. line.

Sol:



Clearly ans is (d) because there is a property of parabola which states the locus of point of intersection of the mutually perpendicular tangents to a parabola is directrix of the parabola.

Eq. of tangent at P: $ty = x + at^2$ ...(i)

" " " Q: $sy = x + as^2$ ...(ii)

Point of intersection of these tangents by equating (i), (ii) is $(ats, a(t+s))$. Let this point be (h,k).

$\left.\begin{array}{l} h = ats \\ k = a(t+s) \end{array}\right\}$ Slope of tangents are $\frac{1}{s}$ and $\frac{1}{t} \Rightarrow \frac{1}{t}\times\frac{1}{s} = -1 \Rightarrow ts = -1$

$\therefore h = -a, \ k = 0$

$\Rightarrow h = -a \Rightarrow x = -a \Rightarrow x + a = 0$ (Eq of directrix of parabola).

11) Let $f: R \to R$ be given by $f(x) = |x^2-1|$, $x \in R$

Then, a) f has a local minima at $x = \pm 1$ but no local max.

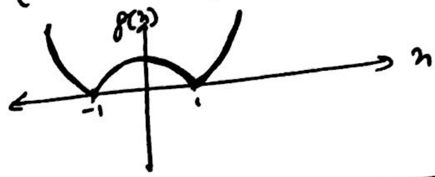
b) f has a local maximum at $x=0$ but no local minima

✓c) f has a local minima at $x=\pm 1$ and a local max at $x=0$

d) none of above is true.

Sol: $f(x) = |x-1||x+1| = (x-1)(x+1)$, $x \geq 1$
$= -(x-1)(x+1)$, $-1 \leq x < 1$
$= (x-1)(x+1)$, $x < -1$

$$\therefore f'(x) = \left.\begin{aligned} &2x, && x \geq 1 \\ &-2x, && -1 \leq x < 1 \\ &2x, && x < -1 \end{aligned}\right\} \Rightarrow \begin{aligned} &f'(1^+) = 2 \;;\; f'(-1^+) = 2 \\ &f'(1^-) = -2 \;;\; f'(-1^-) = -2 \end{aligned}$$



option c)

12) The no of triplets (a,b,c) of positive integers satisfying $2^a - 5^b 7^c = 1$ is a) infinite b) 2 c) 1 ✓d) 0

Sol: $2^a - 5^b 7^c = 1$ (i), Min value of $b, c = 1 \Rightarrow a \geq 5$

Dividing equation by 8, we get

$0 - (5 \text{ or } 1)(-1)^c \equiv 1 \pmod 8$ (ii) [$\because$ If b is odd, $5^b \equiv 5 \pmod 8$; If b is even, $5^b \equiv 1 \pmod 8$]

clearly to hold the congruence relation, b must be even & c must be odd.

Now dividing (i) by 5. $\therefore 2^a - 0 \equiv 1 \pmod 5$.

$\Rightarrow a$ is divisible by 4 because $(2^2)^{2m} \equiv (-1)^{2m} \equiv 1 \pmod 4$

Now dividing equation by 3 we get,

$(-1)^a - (-1)^b (1)^c \equiv 1 \pmod 3$

As a, b are even, $\therefore 1 - 1 \equiv 1 \pmod 3$ (Impossible). Hence option (d).

13) Let a be a fixed ~~point~~ real no. greater than -1. The locus of $z \in C$, satisfying $|z - ia| = \text{Im}(z) + 1$ is

✓a) parabola b) ellipse c) hyperbola d) not a conic.

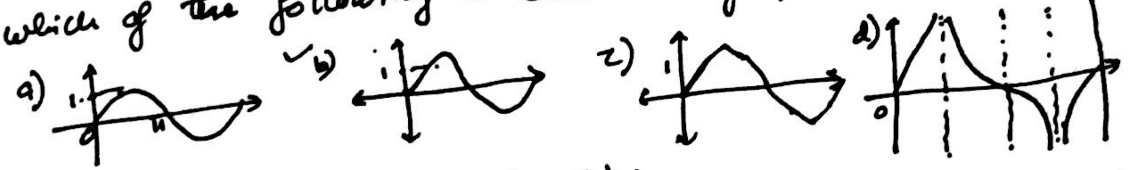
Sol: Let $z = h + ik$

$\therefore |h+ik - ia| = k+1 \Rightarrow \sqrt{h^2+(k-a)^2} = k+1$

$\Rightarrow h^2+(k-a)^2=(k+1)^2 \Rightarrow h^2 - 2k(a+1) + a^2-1=0$

$\Rightarrow x^2 - 2y(a+1) + a^2-1 = 0 \Rightarrow$ Locus is parabola.

14) Which of the following is closest to graph $\tan(\sin x)$, $x>0$?



Sol: i) $\tan(1) > \tan(\frac{\pi}{4}) \Rightarrow \tan(\sin\frac{\pi}{2}) > 1$

$\Rightarrow \tan(\sin x) > 1 \Rightarrow$ (option a not possible

ii) $\tan(\sin x)$ is a curve, not discrete line. (option c not possible)

iii) $\because -1 < \sin x < 1 \therefore \tan(\sin x)$ is continuous for $x \in R$

$\therefore$ option (b).

15) Consider the function $f: R\setminus\{1\} \to R\setminus\{2\}$ given by $f(x) = \frac{2x}{x-1}$

Then, a) f is one-one but not onto

b) f is onto but not one-one

c) f is neither one-one nor onto

✓ d) f is both one-one & onto.

Sol: Let $f(x_1) = f(x_2)$

$\frac{2x_1}{x_1-1} = \frac{2x_2}{x_2-1} \Rightarrow \frac{x_1}{x_2} = \frac{x_1-1}{x_2-1} \Rightarrow \frac{x_1-x_2}{x_2} = \frac{x_1-x_2}{x_2-1}$

$\Rightarrow (x_1-x_2)\left[\frac{1}{x_2} - \frac{1}{x_2-1}\right] = 0 \Rightarrow (x_1-x_2)\left[\frac{1}{1-x_2}\right] = 0$

$\because \frac{1}{1-x_2} \neq 0 \Rightarrow x_1 = x_2$ (Injective).

Let $y = \frac{2x}{x-1} \Rightarrow yx - y = 2x \Rightarrow (y-2)x = y \Rightarrow x = \frac{y}{y-2}$

$\therefore$ Range is $R-\{2\}$. = Codomain. Hence option d.

16) Consider a real valued function continuous on $x \in R$ satisfying $f(x+1) = f(x)$. Let $g(t) = \int_0^t f(x)\,dx$, $t \in R$

Define $h(t) = \lim_{n\to\infty} \frac{g(t+n)}{n}$, provided the limit exists. Then

a) $h(t)$ is defined only for $t=0$

b) $h(t)$ is defined only when t is an integer

✓ c) $h(t)$ is defined for all $t \in R$ & is independent of t.

d) None of above is true.

Now $g(t) = \int_0^t f(x)\,dx = (t-0) f(c) = t \cdot f(c)$

Mean Value of function over an interval
$\int_a^b f(x)\,dx = (b-a) \cdot f(c)$ where $c \in (a,b)$.

$g'(t) = f(c)$

$\because f(x+1) = f(x) \Rightarrow f(c) = f(c+1) \Rightarrow g'(t) = g'(t+1)$

Now $g'(t+n) = g'(t+n-1) = g'(t+n-2) = \dots g'(t) = f(c)$

$h(t) = \lim_{n\to\infty} \frac{g(t+n)}{n} \left(\frac{\infty}{\infty}\right) = g'(t+n) = f(c)$

Hence option C.

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17) Consider the sequence $a_1 = 24^{1/3}$, $a_{n+1} = (a_n + 24)^{1/3}$, $n \geq 1$

Then the integer part of a_{100} =

✓a) 2 b) 10 c) 100 d) 24

Sol: $a_1 = 24^{1/3}$; $a_2 = \sqrt[3]{a_1+24}$, $a_3 = \sqrt[3]{a_2+24} = \sqrt[3]{24+\sqrt[3]{24+24^{1/3}}}$

$= \sqrt[3]{24^{1/3}+24}$

$a_n = \sqrt[3]{24+\sqrt[3]{24+\sqrt[3]{24+\cdots}}}$; if $n\to\infty$, $a_n = \sqrt[3]{24+a_n}$

if $n\to\infty$ $a_n^3 = 24 + a_n \Rightarrow a_n^3 - a_n = 24$

$\Rightarrow a_n(a_n^2 - 1) = 2\times3\times4 \Rightarrow (a_n-1)(a_n)(a_n+1) = 2\times3\times4$

$\therefore a_n - 1 = 2$, $a_n = 3$, $a_n+1 = 4$

This mean if $n\to\infty$, $a_n = 3$; so if $n\to 100$ $a_n < 3$

Now we have $a_{n+1} = (a_n+24)^{1/3}$

we have $a_n < 3 \Rightarrow a_n + 24 < 27 \Rightarrow (a_n+24)^{1/3} < 3$

$\Rightarrow a_{n+1} < 3 \Rightarrow a_{100} < 2$ $\therefore$ integer part of a_{100} is 2.

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18) Let $(x,y) \in (-2,2)$ and $xy = -1$. Then minimum value of $\frac{4}{4-x^2} + \frac{9}{4-y^2}$ is a) 8/5 b) 12/5 c) 12/7 d) 15/7

Sol: $f(x) = \frac{4}{4-x^2} + \frac{9}{4-y^2}$ $\left(\because y = -1/x\right)$ $\therefore f(x) = \frac{4}{4-x^2} + \frac{9}{9-\frac{1}{x^2}}$

$= \frac{4}{4-x^2} + \frac{9x^2}{9x^2-1} = \frac{4}{4-x^2} + \frac{1}{9x^2-1} + 1 = \frac{35x^2}{(4-x^2)(9x^2-1)} + 1$

$f'(x) = 0 \Rightarrow \frac{35\left[\{2x(4-x^2)(9x^2-1) - x^2\{-2x(9x^2-1) + (4-x^2)18x\}\right]}{(4-x^2)^2(9x^2-1)^2} = 0$

$\Rightarrow 9x^4 - 4 = 0 \Rightarrow x^2 = 2/3$

+ $-\sqrt{2/3}$ − $\sqrt{2/3}$ +

$\therefore f(x)_{min}$ at $x = \pm\sqrt{2/3} = \frac{35\times 2/3}{(4-2/3)(9\times2/3-1)} = 12/5$.

Gr (7)